Selected Solution to Final Exam

Answer all six questions.

1. Let f be a C^1-function in $\mathbb{R}^3$.

   (a) (10 points) Suppose that $f(x_0, y_0, z_0) = 0$ and $f_z(x_0, y_0, z_0) \neq 0$. Using the inverse function theorem to prove that there is an open disk D containing (x_0, y_0) and a C^1-function φ on D satisfying $\varphi(x_0, y_0) = z_0$ such that $f(x, y, \varphi(x, y)) = 0$ for all $(x, y) \in D$.

   (b) (5 points) Let $Z(f) = \{(x, y, z) : f(x, y, z) = 0\}$ be the zero set of f. Assume that its gradient is nonvanishing, that is, $\nabla f \neq (0, 0, 0)$ everywhere in $\mathbb{R}^3$. Show that $Z(f)$ is a nonwhere dense set in $\mathbb{R}^3$.

   (c) (5 points) Can we find a sequence of C^1-functions $\{f_k\}$ with non-vanishing gradient everywhere such that $\mathbb{R}^3 = \bigcup_{k=1}^{\infty} Z(f_k)$?

   Solution. (a) See Notes.

   (b) First of all, clearly $Z(f)$ is a closed set (as long as f is continuous.) Let (x, y, z) be a point on $Z(f)$. WLOG we may assume $f_z \neq 0$ at this point, so near this point $Z(f)$ is given by the graph $(x, y, \varphi(x, y))$. It follows that for small $\varepsilon \neq 0$, the points $(x, y, \varphi(x, y) + \varepsilon)$ do not belong to $Z(f)$. It shows that $Z(f)$ cannot contain a ball in $\mathbb{R}^3$. Hence it is nowhere dense.

   (c) This is a direct consequence os Baire's category theorem, since each $Z(f_k)$ is closed and nowhere dense, and on the other hand, $\mathbb{R}^3$ is complete.

2. (a) (10 points) State the theorem on the perturbation of identity.

   (b) (10 points) Use (a) to show that the equation $x \sin x - x^4 + x = -0.02$ has a root near $x = 0$.

   Solution. See Notes.

3. Consider the IVP: $x' = F(t, x),\ x(0) = 0$, where F is a nonnegative, continuous function in $\mathbb{R}^2$ satisfying the Lipschitz condition $|F(t, x_2) - F(t, x_1)| \leq L|x_2 - x_1|,\ (t, x_1), (t, x_2) \in \mathbb{R}^2$.

   (a) (10 points) Show that if $x(t)$ is a solution of this IVP on $[0, c)$ for some finite $c > 0$, then it can be extended to be a solution on $[0, c]$ unless $x(t) \uparrow \infty$ as $t \uparrow c$.

   (b) (10 points) Assume further that $F(t, x) \leq C(1 + x), \forall (t, x) \in \mathbb{R}^2$, for some constant C. Show that this IVP admits a solution in $[0, \infty)$.

   Solution. (a) See Notes. Or, as we have extra assumption F is nonnegative, the argument can be simplified as follows: From $x' = F \geq 0$ we see that x is increasing. Hence, $z = \lim_{t \to c^-} x(t)$ always exists unless x does not tend to ∞ at c. Extend x from $[0, c)$ to $[0, c]$ by defining $x(c) = z$. By letting $t \to c^-$ in the relation

   $$x(t) = \int_0^t F(s, x(s))\, ds\ , \quad t \in [0, c),$$

   the LHS tends to $z = x(c)$ and the RHS tends to $\int_0^c F(s, x(s))\, ds$. Hence

   $$x(c) = z = \int_0^c F(s, x(s))\, ds,$$

so x solves the equation also at $t = c$.

Solution Integrating the relation $x' \le C(1+x)$ we get $x(t) \le e^{Ct} - 1$ as long as the solution exists. Let $c^* = \sup\{c : x \text{ exists on } [0,c)\}$. By uniqueness of IVP, the solution x exists on $[0,c^*)$. If c^* is finite, by (a) it extends to be a solution on $[0,c^*]$. Then by solving the IVP taking c^* as the initial time, x extends beyonds c^*, contradicting the definition of c^*. We conclude that c^* is ∞, done.

4. (10 points) Show that $\{\cos nx : x \in [0,2\pi],\ n \ge 1\}$ does not have any convergent subsequence in $\|\cdot\|_\infty$.

   Solution. Suppose on the contrary there is convergent subsequence $\{\cos n_k x\}$ we want to draw a contradiction. Since $\{\cos n_k x\}$ is convergent, it is precompact. By Arzela's theorem this subsequence is also equicontinuous. For $\varepsilon = 1/2$, there is some δ such that $|\cos n_k x - \cos n_k y| < 1/2$ wherever $|x-y| < \delta$ for all sufficiently large n_k. Now, take $x = 0$ and $y = \pi/(2n_k)$. For large n_k, $|y - 0| < \delta$, but $|\cos n_k 0 - \cos n_k \pi/(2n_k)| = 1 > 1/2$, contradiction holds.

5. Let $K \in C([a,b] \times [a,b])$ and define the operator T by

$$(Tf)(x) = \int_a^b K(x,y) f(y) dy.$$

   (a) (10 points) Show that T maps $C[a,b]$ to itself.

   (b) (10 points) Show that whenever $\{f_n\}$ is a bounded sequence in $C[a,b]$, $\{Tf_n\}$ contains a convergent subsequence in the sup-norm.

   Solution. See Exercise.

6. (10 points) Show that there exists a unique nonnegative solution h to the integral equation

$$h(x) = 1 + \frac{1}{2}\int_0^1 \frac{1}{1+x+y} h(y)\, dy,$$

   in $C[0,1]$. Suggestion: Work on the space $X = \{h \in C[0,1] : h(x) \ge 0\}$.

   Solution. Define

$$Th(x) = 1 + \frac{1}{2}\int_0^1 \frac{1}{1+x+y} h(y)\, dy$$

   and verify it is a contraction. However, in order to apply the contraction mapping principle, you need to explain X is a complete metric space under the sup-norm, and (b) $Th \in X$. Many of you forgot to point out $Th \in C[0,1]$.